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Extended summary

Considerations on the Static-Kinematic Duality
Theoretical Models and Numerical Applications

Curriculum: Architecture, Buildings and Structures

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Abstract. The last decades have been involved in a rapid development of computer science and electronics with major worldwide scientific and technical consequences. In structural mechanics the variational energetic principles have become the theoretical basis of many numerical analysis procedures.
The thesis is a study on the Principle of Virtual Work and its practical use to deduce structural and theoretical models in various fields. Particular emphasis is also placed on the numerical approach to these models, since the use of personal computers in any engineering application has become very common.

The main aim reached is definitely to have shown that even at first sight very complex structural theories, and related numerical analysis methods, are actually based on a simple principle whose utterance occupies a few lines, remaining in the specific only to define the

environment in which this principle operates and the implementation on a computer of a good solver algorithm.

As original examples of practical application of the PVW has been proposed:

- An analysis method for pin-jointed structures, practically used to study a prestressed pin-jointed steel structure.
- A second order kinematics for curved beams, of which has also been given a physical and geometrical explanation.
- A threedimensional Cosserat-like model to define the behavior of ferroelectrics materials, together with its physical interpretation at atomic level.
- A generic method to transform a monodimensional continuous model in a discrete one, through the definition of suitable finite elements, usable for each non linear second order kinematics.
- A PVW generalization in dynamics, which involves the notion of duality even when come into play the time dimension.

Keywords. Principle of Virtual Work, Numerical Methods, Theoretical Models

1 Problem statement and objectives

The Principle of Virtual Work can be stated as "*A configuration is in equilibrium if and only if the virtual external work is equal to the virtual internal work, for all admissible virtual displacements*".
The mathematical model offered to a university student in the first continuum mechanics course to describe the behaviour of a three-dimensional Cauchy elastic continuum is based upon the ideas of equilibrium and compatibility. Then, with the aid of the constitutive law, the equations of the elastic problem are deduced. Once defined compatibility and equilibrium, it is shown that there is a duality between them and is stated the PVW, usually proposed as a theorem. In a similar way is deduced the elastic line of a beam, defining the bending moment, the curvature and the stiffness EJ as the constitutive relation.
The reason of such approach is very simple: the PVW is difficult to be understood by a student, which is more comfortable in visualizing a stress tensor with a clear physical meaning, and to associate to it a second tensor, the strain one, with the same structure and correlated with the first through a constitutive law. Only familiarized himself with these two notion the PVW is stated and demonstrated.
Actually, if compatibility and equilibrium are independently defined, the PVW is not necessarily satisfied. The main task of structural mechanics is to provide a good model to describe a structure, either if we are going to study a beam, a plate, a ferroelectric material, etc... The list can be very long, both for discrete structures and continuous ones in one, two or three dimension. The equations written for each of this model may be very various, but they always satisfy the PVW. As stated by its name, is therefore preferable to accept the PVW as an axiom and as a necessary condition to validate a mechanical model.
A consequence of the PVW is that, if the system has a potential, the equilibrium is reached in stationary conditions of this potential. Nevertheless the PVW has a wider validity, and can be applied also in non-conservative ambit (for example in models with non symmetric constitutive law).
The equation of the PVW may be written as

$$\mathbf{f} \cdot \underline{\mathbf{d}} = \boldsymbol{\sigma} \cdot \underline{\boldsymbol{\varepsilon}} \qquad (1)$$

In this equation $\mathbf{f}$ represents the external forces, $\mathbf{d}$ is the displacements vector, $\boldsymbol{\sigma}$ are internal stresses and $\boldsymbol{\varepsilon}$ are the strains. The product is intended as a scalar product, and the underline means that the quantity is virtual. The left-hand side is therefore the virtual external work, while the right-hand one is the virtual internal work. Actually we should complete the equation with integration symbols for continuous structures, and we should add external boundary forces. For discrete structures we should use the summation symbols over nodes and elements. In finite elements formulations, in which a function is written as a finite linear combination of basic shape functions, appears both the summation and integration symbols.
Eq. (1) reveals perfectly the Static-Kinematic Duality. There are two couples of entities, which do mutual work: forces do work for displacements, and internal stresses do work for strains. We remark that the concept of forces and stresses are completely generalized: a force is any object which do work for a degree of freedom (displacement) of the structure, while a stress is any object which do work for a strain.

We can clarify the meaning of the equation by explicating the dependence on **d**. The equation as in fact **d** as unknown, and to be solved we have to express all quantities in terms of **d**:

$$\mathbf{f}(\mathbf{d}) \cdot \underline{\mathbf{d}} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon}(\mathbf{d})) \cdot \underline{\boldsymbol{\varepsilon}}(\mathbf{d}) \qquad (2)$$

External forces **f**, in a general treatment, depend on displacements **d** (live loads). The stresses depend **σ** depend on **ε** through the constitutive law. In linear elasticity this relation is represented by a square matrix, symmetric if the material is conservative. The usage of a matrix to represent the constitutive law is frequent even in non linear models. We have in this case a geometric non-linearity, since non linear terms come from the compatibility.
We can also suppose that there is an initial prestress. A structure is prestressed if in the reference configuration (**d**=0) we have **σ**≠0.
The compatibility (or kinematics) expresses the strain **ε** in terms of displacement **d**. In discrete linear models this relation is represented by a matrix. In continuous model the strains depend not only on **d**, but also on the derivatives of **d**. In non linear models is often sufficient to express the stresses as a second order polynomials in **d** (second order kinematics). This assumption is usually sufficient to study the structure, even if it is instable (post-bucking analysis) or if it is prestressed.
Virtual strains **ε̲** is always linear in virtual displacements **d̲**, since a quantity is virtual if it belongs to the tangent bundle of its configuration space [1]. In linear models virtual strains depend only on virtual displacements. In second order models, virtual strains are linear both in the displacements **d** and in virtual displacements **d̲**.
Using the PVW, the equilibrium equation can be deduced from the kinematics (and vice versa). The equilibrium is a relation between external forces and stresses. In kinematically non linear models the equilibrium equation depend also on the displacements, since we need to write the equilibrium in the deformed configuration. Finally if we express internal stresses by terms of displacements, we obtain the equilibrium written a la Navier, which is a direct relation between forces and displacements.
These equations can be directly solved. The procedure is simple for discrete structure, or for structure discretized using finite elements: since the PVW is valid for each admissible virtual displacement, **d̲** can be simplified, obtaining an equation in **d**.
For continuous structure the procedure consists in applying the integration by part formula (for monodimensional structures) or the Gauss-Green formula (bi- and three-dimensional structures). This allow to remove the differential operators from the virtual displacements, so that they can be simplified.

2 Research planning, activities, analysis and discussion of main results

The thesis is an excursion into the PVW and into its application in deducing theoretical models and their numerical implementations. It is organized as follows.

First chapter
The first chapter briefly introduces the PVW and the duality concept which is at its base. It defines the theoretical background from which the thesis is developed.

Second chapter
The second chapter analyzes the behaviour of discrete structures. A structure is called discrete if its state is defined by a finite number of degrees of freedom (displacements). Assuming a general second order kinematics and including constrains effects, a general theory for discrete structures is derived. A second order kinematics is often sufficient to study the behaviour of a structure, even in relation to post-buckling and in presence of initial prestress.
This general theory is then specialized for pin jointed prestressed structures, also known as Tensegrities. The classic approach to Tensgrities [3],[4], consists in writing the equilibrium equations in the deformed configuration. Here we directly start from the definition of the elongation of bars as a strain, and we use the PVW to deduce the equilibrium. As a practical application of the deduced theory, it is included a preliminary study of a pin-jointed tensegrity structure in glass and steel, which is going to be the covering of the main hall in a major Italian museum. This study has been done in MATLAB environment [11].

Third chapter
The third chapter is about monodimensional continuous structures. As for discrete structure, it is proposed a general model within the assumption of a second order kinematics. This general theory is applied to deduce the kinematics of curved rods, both unshearable and with shearing deformations. The approach consist in writing the equilibrium equations in the deformed configuration, including non-conservative effects of rotations [23] and then deducing the kinematics.
We give then physical interpretation of second order terms appearing in the kinematics, especially in relation to buckling phenomena. Also we discuss the linear effects of initial curvature in a rod. An interesting result of the study is the description and explanation of a particular case of instability due to traction.
At the end of the chapter a similar approach is used to define the finite kinematics of curved rods [14][15].

Fourth Chapter
In the fourth chapter is proposed a threedimensional Cosserat-like model [30] to describe ferroelectrics behaviour, especially in relation to domain wall formation. A simple available monodimensional model [33], if generalized in three dimension, leads to results which are incompatible with the available literature on ferroelectrics [31]. Hence we notice that the variational model [33] is mathematically equivalent to a beam on elastic soil.
A natural refinement to this model is the shearable beam. Indeed its threedimensional generalization is the Cosserat model, and it is compatible with recent results for ferroelectrics. In particular, the Cosserat micropolar shear energy has the same mathematical expression of the depolarization energy of ferroelectrics [29].
The deduced kinematics can be explained at atomistic level, and it is interesting to notice that the analogy between the Cosserat model and the ferroelectric behaviour is only mathematical: in ferroelectrics the roles of rotation and displacements are swapped in comparison with Cosserat theory. Nevertheless the resulting equations are similar.
This chapter includes a numerical analysis (in MATLAB) of the behaviour of a shearable beam on elastic soil using finite elements, and hence is also a link between the third chapter (kinematics of rods) and the second one (discrete structure). The methodology used is care-

fully described, and can be easily generalized to define the finite element model of any monodimensinal continuum for which is defined a second order kinematics.

Fifth chapter
Finally in the last short chapter we generalize the notion of duality in dynamics, starting from the extended Hamilton principle [43]. The main results of this chapter is that the velocity v can be seen as a strain, and momentum q can be seen as a stress. Since we have

$$q = \varrho\, v \qquad (3)$$

the mass density ϱ can be seen as the constitutive law between them.
The differences between statics and dynamics are the constitutive law and boundary conditions. Constitutive law is not positive definite: this means that we have vibration modes. Also we often have dissipative terms, while in static the behaviour is usually non dissipative.
In dynamics there is also a different way to assign boundary condition, since we give both displacements and velocity at the beginning of the temporal interval, obtaining a Cauchy problem.
As an example an old model about axial behaviour of viscous bars [42] is reformulated using duality concepts.

3 Conclusions

The main aim reached is definitely to have shown that even at first sight very complex structural theories, and related numerical analysis methods, are actually based on a simple principle whose utterance occupies a few lines, remaining in the specific only to define the environment in which this principle operates and the implementation on a computer of a good solver algorithm.
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